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Path Magic Graphs

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Abstract. Let $G = (V,E)$ be a graph with $|V| = p$ and $|E| = q$. A total labeling $\lambda : V \cup E \to \{1,2,\ldots,p+q\}$ is a bijection. The weight $\lambda^* : V \cup E \to N$ is a function induced by λ such that $\lambda^*(v) = \lambda(v) + \sum_{x\sim v} \lambda(vx)$ and $\lambda^*(e) = \lambda(u) + \lambda(v) + \lambda(e)$, where $e = uv$. If $\lambda^*(v)$ is a constant for all $v \in V$, then G is called vertex-magic. If $\lambda^*(e)$ is a constant for all $e \in E$, then G is said to be edge magic. If G is both vertex magic and edge magic, G is called a totally magic graph. In this paper, we define path magic graphs and prove that any star graph is path magic. We also prove that totally magic graphs are path magic, but not the converse.

Keywords: P_n-magic, path magic, A-labeling.

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1 Introduction

Magic labelings have been studied by many authors, for example, vertex magic labeling by Daisy Cunningham [4] and MacDougall [9] and edge magic labeling by Baskaro [1]. An account of various types of magic labelings can be found in [7].

Let $G = (V,E)$ be a graph with $|V| = p$ and $|E| = q$. A total labeling $\lambda : V \cup E \to \{1,2,\ldots,p+q\}$ is a bijection. The weight $\lambda^* : V \cup E \to N$ is a function induced by λ such that $\lambda^*(v) = \lambda(v) + \sum_{x\sim v} \lambda(vx)$ and $\lambda^*(e) = \lambda(u) + \lambda(v) + \lambda(e)$, where $e = uv$. If $\lambda^*(v)$ is a constant for all $v \in V$, then G is called vertex-magic. If $\lambda^*(e)$ is a constant for all $e \in E$, then G is said to be edge magic. If G is both vertex magic and edge magic, G is called a totally magic graph.

Definition 1.1. *Let λ be a total labeling of a graph $G = (V,E)$ with v vertices and e edges. The weight of a path P_n is defined as the sum of the weight of all the vertices and edges in the path and is denoted by* $\mathrm{wt}_\lambda(P_n)$*. If the weight of all the n-vertex paths P_n is a constant for some n $(n \geq 2)$, then G is called P_n-magic. If G is P_n-magic for all $n \geq 2$, then G is said to be path magic and λ is called a path magic labeling of G.*

In [6], Exoo has defined totally magic graphs and has proved in Corollary 6.1 that the only totally magic star is $K_{1,2}$. In Section 2, we prove that any star $K_{1,n}$ is path magic. In [2], Combe has studied A-magic labeling where the labels are the elements of a finite abelian group of order $p+q$. He has proved in Theorem 14 that any star with more than four rays has vertex magic A-labelings and edge magic A-labelings for all choices of abelian groups A of appropriate order. In Section 3, we prove that any star $K_{1,n}$ is path magic over any cyclic group of order $2n+1$. For terminology and symbols, we refer to [8].

2 Main results

Theorem 2.1. *A star graph $K_{1,n}$ $(n \geq 1)$ is path magic.*

Proof. Let u be the central vertex and u_i $(1 \leq i \leq n)$ be the pendant vertices. Define $\lambda : V \cup E \to \{1, 2, \ldots, 2n+1\}$ by

$$\lambda(u) = 1$$
$$\lambda(u_i) = i+1, \quad 1 \leq i \leq n$$
$$\lambda(uu_i) = 2n - (i-2), \quad 1 \leq i \leq n.$$

Clearly λ is a bijection.

$$\begin{aligned} \mathrm{wt}_\lambda(u) = \lambda(u) + \sum_{i=1}^{n} \lambda(uu_i) &= 1 + \sum_{i=1}^{n} [2n - (i-2)] \\ &= 1 + \frac{3}{2}n(n+1) \end{aligned}$$

Similarly, for $1 \leq i \leq n$, $\mathrm{wt}_\lambda(u_i) = 2n+3$ and $\mathrm{wt}_\lambda(uu_i) = 2n+4$.
For any 2-path $P_2(u_iu)$,

$$\begin{aligned} \mathrm{wt}_\lambda(P_2(u_iu)) &= \mathrm{wt}_\lambda(u_i) + \mathrm{wt}_\lambda(u_iu) + \mathrm{wt}_\lambda(u) \\ &= (2n+3) + (2n+4) + \left[1 + \frac{3}{2}n(n+1)\right] \\ &= \text{constant.} \end{aligned}$$

For any 3-path $P_3(u_iuu_j)$,

$$\begin{aligned} \mathrm{wt}_\lambda(P_3(u_iuu_j)) &= \mathrm{wt}_\lambda(u_i) + \mathrm{wt}_\lambda(u_iu) + \mathrm{wt}_\lambda(u) + \mathrm{wt}_\lambda(uu_j) + \mathrm{wt}_\lambda(u_j) \\ &= 2(2n+3) + 2(2n+4) + \left[1 + \frac{3}{2}n(n+1)\right] \\ &= \text{constant.} \end{aligned}$$

Therefore $K_{1,n}$ is path magic. $\square$

Theorem 2.2. *Any totally magic graph is path magic.*

Proof. Let λ be a totally magic labeling of G with vertex constant and edge constant h and k respectively. For any n-path P_n in G, $\mathrm{wt}_\lambda(P_n) = nh + (n-1)k =$ constant.

Here G is path magic. □

Remark 2.3. *The converse of the above theorem is not true. For example, $K_{1,n}$ $(n \geq 3)$ is path magic, but not totally magic as in Corollary 6.1 in [6].*

Theorem 2.4. *A P_2-magic labeling of G which is also vertex magic is totally magic.*

Proof. Let λ be both P_2-magic and vertex magic with vertex constant h.

Since λ is P_2-magic, for any edge uv in G, $\mathrm{wt}_\lambda(P(uv)) = k$, a constant.

That is, $\mathrm{wt}_\lambda(u) + \mathrm{wt}_\lambda(uv) + \mathrm{wt}_\lambda(v) = k$.

$\mathrm{wt}_\lambda(uv) = k - 2h$, a constant.

Therefore λ is edge magic and hence totally magic. □

Corollary 2.5. *A vertex magic labeling o f G is P_2-magic if and only if it is totally magic.*

Proof. Follows from Theorems 2.4 and 2.2. □

In [4], Daisy Cunningham has proved that every cycle C_n admits a vertex magic total labeling. In view of this, we have the following

Theorem 2.6. *If λ is a vertex magic total labeling of C_n $(n \geq 3)$, then λ cannot be P_2-magic.*

Proof. Suppose λ is P_2-magic, then by Theorem 2.4, λ is totally magic, which is not possible as in Theorem 5.2 in [6]. □

The existence of a vertex magic total labeling of C_n leads to the existence of an edge magic total labeling of C_n as seen below.

Theorem 2.7. *Every vertex magic total labeling of C_n gives rise to an edge magic total labeling of C_n and vice versa.*

Proof. Let λ be a vertex magic total labeling of $C_n : u_1u_2\dots u_nu_1$ with vertex constant h.

Define $\lambda' : V \cup E \to \{1,2,\dots,2n\}$ by

$\lambda'(u_i) = \lambda(u_{i-1}u_i)$, $2 \le i \le n+1$

$\lambda'(u_iu_{i+1}) = \lambda(u_i)$, $1 \le i \le n$ with the convention that $u_{n+1} = u_1$.

$$\begin{aligned}\mathrm{wt}_{\lambda'}(u_iu_{i+1}) &= \lambda'(u_i) + \lambda'(u_iu_{i+1}) + \lambda'(u_{i+1}) \\ &= \lambda(u_{i-1}u_i) + \lambda(u_i) + \lambda(u_iu_{i+1}) \\ &= \mathrm{wt}_\lambda(u_i) = h, \quad 1 \le i \le n.\end{aligned}$$

That is, λ' is edge magic. □

The proof of the converse part is similar.

Theorem 2.8. *Let λ be a P_2-magic labeling of G which is also edge magic. Then G cannot be an odd cycle C_n $(n > 3)$.*

Proof. Suppose $G = C_{2m+1}$, $m \ge 2$.

Let C_{2m+1} be the cycle $u_1u_2\cdots u_{2m+1}u_1$. Since λ is P_2-magic, $\mathrm{wt}_\lambda(P_2(u_1u_2)) = \mathrm{wt}_\lambda(P_2(u_2u_3))$, which implies $\mathrm{wt}_\lambda(u_1) = \mathrm{wt}_\lambda(u_3)$, since λ is edge magic. Similarly considering the paths $P_2(u_2u_3)$ and $P_2(u_3u_4)$; $P_2(u_3u_4)$ and $P_2(u_4u_5)$; $\dots$; $P_2(u_{2m-1}u_{2m})$ and $P_2(u_{2m}u_{2m+1})$; $P_2(u_{2m}u_{2m+1})$ and $P_2(u_{2m+1}u_1)$, we get,

$$\mathrm{wt}_\lambda(u_1) = \mathrm{wt}_\lambda(u_2) = \cdots = \mathrm{wt}_\lambda(u_{2m}) = \mathrm{wt}_\lambda(u_{2m+1}).$$

That is, C_{2m+1} is vertex magic and hence totally magic, which is a contradiction according to Corollary 5.2 in [6].

Hence the theorem. $\square$

Theorem 2.9. *The tree $B_{m,n}$ is P_4-magic.*

Proof.

$$\begin{aligned} \text{Let} \quad & V(B_{m,n}) = \{u, v, u_i, v_j : 1 \le i \le m, 1 \le j \le n\}. \\ \text{Then} \quad & E(B_{m,n}) = \{uu_i, vv_j, uv : 1 \le i \le m, 1 \le j \le n\} \end{aligned}$$

Define $\lambda : V \cup E \to \{1, 2, \ldots, 2m+2n+3\}$ by

$$\lambda(uu_i) = i, \quad 1 \le i \le m$$

$$\lambda(vv_j) = m + j, \quad 1 \le j \le n$$

$$\lambda(v_j) = m + 2n + 1 - j, \quad 1 \le j \le n$$

$$\lambda(u_i) = 2m + 2n + 1 - i, \quad 1 \le i \le m$$

$$\lambda(u) = 2m + 2n + 1, \quad \lambda(uv) = 2m + 2n + 2, \quad \lambda(v) = 2m + 2n + 3.$$

Clearly, λ is a bijection.

It can be found that

$$\begin{aligned} \mathrm{wt}_\lambda(u) &= 4m + 4n + 3 + \frac{m(m+1)}{2}, \\ \mathrm{wt}_\lambda(v) &= 4m + 4n + 5 + mn + \frac{n(n+1)}{2}, \\ \mathrm{wt}_\lambda(u_i) &= 2m + 2n + 1 \quad 1 \le i \le m \\ \mathrm{wt}_\lambda(v_j) &= 2m + 2n + 1 \quad 1 \le j \le n \\ \mathrm{wt}_\lambda(uu_i) &= 4m + 4n + 2 \quad 1 \le i \le m \end{aligned}$$

$$\mathrm{wt}_\lambda(vv_j) = 4m+4n+4 \quad 1 \le j \le n$$
$$\mathrm{wt}_\lambda(uv) = 6m+6n+6$$

Any 4-path P_4 in $B_{m,n}$ is of the form u_iuvv_j, $1 \le i \le m$, $1 \le j \le n$ and

$$\begin{aligned}\mathrm{wt}_\lambda(P_4) &= \mathrm{wt}_\lambda(u_i) + \mathrm{wt}_\lambda(u_iu) + +\mathrm{wt}_\lambda(u) + \mathrm{wt}_\lambda(uv) + \mathrm{wt}_\lambda(v) + \mathrm{wt}_\lambda(vv_j) + \mathrm{wt}_\lambda(v_j)\\ &= \text{constant.}\end{aligned}$$

Hence $B_{m,n}$ is P_4-magic. □

3 Labeling over a finite abelian group

In [2], Combe has defined total A-labeling of a graph $G = (V,E)$ with p vertices and q edges as a bijection $\lambda : V \cup E \to A$, where A is an abelian group of order $p+q$.

Definition 3.1. *Let G be a graph with p vertices and q edges and A be an abelian group of order $p+q$. A total A-labeling of λ of G is said to be P_n-magic $(n \ge 2)$ if for any n-path P_n in G, $\mathrm{wt}_\lambda(P_n)$ is a constant. If λ is P_n-magic for all $n \ge 2$, then λ is called a path magic labeling of G over A and G is said to be path magic over A.*

Theorem 3.2. *The star graph $K_{1,n}$ is path magic over Z_{2n+1}.*

Proof. Let $V(K_{1,n}) = \{u, u_i : 1 \le i \le n\}$. Then $E(K_{1,n}) = \{uu_i : 1 \le i \le n\}$.
Define $\lambda : V \cup E \to Z_{2n+1}$ by

$$\lambda(u) = 0; \quad \lambda(u_i) = i, \quad 1 \le i \le n$$
$$\lambda(uu_i) = 2n+1-i, \quad 1 \le i \le n$$

Clearly λ is a bijection.

$$\begin{aligned}
\mathrm{wt}_{\lambda}(u) &= \lambda(u) + \sum_{i=1}^{n} \lambda(uu_i) \pmod{2n+1} = \sum_{i=1}^{n} (2n+1-i) \pmod{2n+1} \\
&\equiv -\sum_{i=1}^{n} i \pmod{2n+1} \\
&\equiv r \pmod{2n+1} \quad \text{where} \quad r \equiv -\sum_{i=1}^{n} i \pmod{2n+1}
\end{aligned}$$

For $1 \leq i \leq n$, $\mathrm{wt}_{\lambda}(u_i) \equiv 0 \pmod{2n+1}$ and $\mathrm{wt}_{\lambda}(uu_i) \equiv 0 \pmod{2n+1}$. It can be easily verified that

$$\mathrm{wt}_{\lambda}(P_2(u_i u)) \equiv r \pmod{2n+1} \quad \text{and} \quad \mathrm{wt}_{\lambda}(P_3(u_i u u_j)) \equiv r \pmod{2n+1}.$$

Hence $K_{1,n}$ is path magic over Z_{2n+1}. □

Corollary 3.3. *$K_{1,n}$ is path magic over any finite cyclic group of order $2n+1$.*

Proof. Since any finite cyclic group of order $2n+1$ is isomorphic to Z_{2n+1}, the corollary follows. □

In [3], Combe has defined translation of an A-labeling λ by $a \in A$ and negative of an A-labeling of G.

Let λ be a total A-labeling of $G = (V, E)$ and $a \in A$. The translation $a + \lambda : V \cup E \to A$ defined by $(a+\lambda)x = a + \lambda(x)$ for $x \in V \cup E$ and $-\lambda : V \cup E \to A$, the negative of λ is defined by $(-\lambda)x = -\lambda(x)$ for $x \in V \cup E$.

Theorem 3.4. *Let $G = (V, E)$ be a finite graph with p vertices and q edges. Let A be an abelian group of order $p+q$. Let λ be a total A-labeling of G and $a \in A$. Then (i) $a + \lambda$ is path magic if and only if λ is path magic (ii) $-\lambda$ is path magic if and only if λ is path magic.*

Proof. (i) $a+\lambda : V\cup E \to A$ is defined by $(a+\lambda)x = a+\lambda(x)$, $x\in V\cup E$. Let P_n be the path $u_1u_2\cdots u_n$, where $n\geq 2$ is arbitrary.

$$\begin{aligned}
\mathrm{wt}_{a+\lambda}(P_n) &= \sum_{i=1}^{n}\mathrm{wt}_{a+\lambda}(u_i)+\sum_{i=1}^{n-1}\mathrm{wt}_{a+\lambda}(u_iu_{i+1})\\
&= \sum_{i=1}^{n}[\mathrm{wt}_{\lambda}(u_i)+a(d(u_i)+1)]+\sum_{i=1}^{n-1}[\mathrm{wt}_{\lambda}(u_iu_{i+1})+3a]\\
&= \sum_{i=1}^{n}\mathrm{wt}_{\lambda}(u_i)+\sum_{i=1}^{n-1}\mathrm{wt}_{\lambda}(u_iu_{i+1})+a(2n-2)+na+3(n-1)a\\
&= \mathrm{wt}_{\lambda}(P_n)+(6n-5)a
\end{aligned}$$

Thus, $\mathrm{wt}_{a+\lambda}(P_n)$ is a constant if and only if $\mathrm{wt}_{\lambda}(P_n)$ is a constant.

That is, $a+\lambda$ is P_n-magic if and only if λ is P_n-magic.

(ii) $-\lambda : V\cup E\to A$ is defined by $(-\lambda)x = -\lambda(x)$, $x\in V\cup E$. $\mathrm{wt}_{-\lambda}(P_n) = -\mathrm{wt}_{\lambda}(P_n)$ for any $n\geq 2$.

Hence $\mathrm{wt}_{-\lambda}(P_n)$ is a constant if and only if $\mathrm{wt}_{\lambda}(P_n)$ is a constant.

That is, $-\lambda$ is P_n-magic if and only if λ is P_n-magic. Since $n\geq 2$ is arbitrary, the theorem follows. □

Example 3.5. *Fig. 3.1 is an Z_9-labeling λ of $K_{1,4}$ defined in Theorem 3.2. Fig. 3.2 exhibits the Z_9-labeling given by* $8+\lambda$.

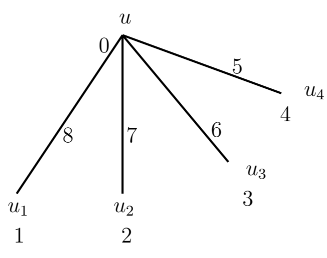

Figure 3.1:

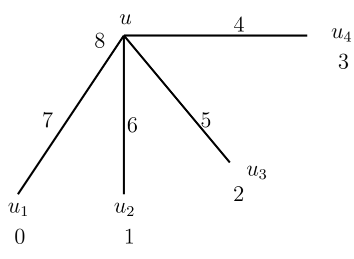

Figure 3.2:

Example 3.6. *Fig. 3.3 is an Z_7-labeling λ of $K_{1,3}$ defined in Theorem 3.2. Fig. 3.4 exhibits the Z_7-labeling given by $-\lambda$.*

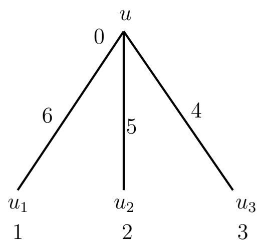

Figure 3.3:

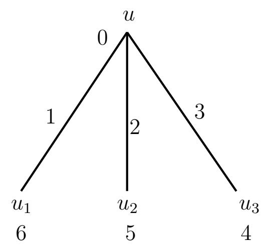

Figure 3.4:

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